ELLIPTIC FUNCTIONS AND PLANE CUBICS

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Abstract. In this paper, I explore elliptic functions, the Weierstrass $\wp$ function and the addition structure on plane cubics.

Contents

1. Elliptic functions

1.1. **The elliptic in elliptic functions.** The starting point of our journey is with mathematicians trying to find a formula for the arc length of an *ellipse*. As contrast to its brother, the circle, it seems quite difficult to find the arc length of an ellipse. For example, on a circle of unit radius, the arc length, measured in the angle from the x-axis, is quite trivially

$$u_c(\varphi) = \int_0^{\varphi} 1 \mathrm{d}\theta = \varphi,$$

but to represent it in the y-coordinate, we can perform the change of variable $y = \sin\varphi, t = \sin\theta$ to get

$$u_c(y) = \int_0^{y} \frac{1}{\sqrt{1-t^2}} \mathrm{d}t = \varphi = \arcsin y$$

which has inverse sin, and note that sin is 2π-periodic.

On the other hand, the same approach when applied onto the "unit" ellipse

$$x^2 + \frac{y^2}{b^2} = 1, \quad b > 1, \quad m \coloneqq 1 - \frac{1}{b^2} > 0$$

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takes us to the following integral, where $r(\theta)$ is the "radius" at angle θ

$$\begin{aligned} u_e(\varphi) &= \int_0^{\varphi} r(\theta)\mathrm{d}\theta \\ &= \int_0^{\varphi} \frac{1}{\sqrt{1-\left(1-\frac{1}{b^2}\right)\sin^2\theta}}\mathrm{d}\theta \\ &= \int_0^{\varphi} \frac{1}{\sqrt{1-m\sin^2\theta}}\mathrm{d}\theta, \end{aligned}$$

then again with the change of variable $y = \sin\varphi, t = \sin\theta$ we get

$$u_e(y) = \int_0^{y} \frac{1}{\sqrt{1-mt^2}} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t,$$

which has inverse sn, the *Jacobi sine function*. As it turns out, similar to sin, sn also has a periodic nature, but now it is *doubly periodic*: it is $4K$- and $2K'i$-periodic where $K, K' \in \mathbb{R}$. We won't delve too much into this, but the point is that the study of *elliptic functions* had its origins in investigating inverses of elliptic integrals, where Fagnano, Euler, Legendre, Abel, Jacobi and others ventured to find addition theorems with a similar flavor to first, for example on the circle, knowing that

$$u_c(\alpha) + u_c(\beta) = u_c(\alpha+\beta),$$

and then reverting to asking about the inverses, yielding

$$\sin(\alpha+\beta) = \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}.$$

1.2. **In the complex analysis setting.** We are interested in the doubly periodic functions, with more flexible periodicity than just along purely real/imaginary offsets like sn. To do so, let's take a step back to investigate periodicity in general on $\mathbb{C}$. f is *ω-periodic* for some $\omega \in \mathbb{C}$ if $f(z) = f(z+\omega)$ for all $z \in \mathbb{C}$. Then for a given f, it is easy to see that the set of ω satisfying this property forms a subgroup in $\mathbb{C}$. They have to be discrete, because otherwise by identity theorem, f has to be constant. However, it might be surprising that there are not that many non-trivial types of such G, only 2.

Proposition 1.1. *There are only 2 types of discrete subgroup $G \neq \{0\}$ of $\mathbb{C}$. Either $G = \mathbb{Z}w$ for some $\omega \neq 0$, or $G = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ for some $\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$.*

Proof. Choose $\omega_1 \in G\setminus\{0\}$ with minimal absolute value. This minimum is achieved because G is a discrete additive group. Since G is an additive group, it follows that $\mathbb{Z}\omega_1 \subseteq G$ too.

If $G = \mathbb{Z}\omega_1$ then we're done. If not, i.e. $G \setminus \mathbb{Z}\omega_1 \neq \varnothing$, again pick $\omega_2 \in G \setminus \mathbb{Z}\omega_1$ with minimal absolute value within $G \setminus \mathbb{Z}\omega_1$, so $\mathbb{Z}\omega_2 \subseteq G$, but G is additive so $\{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \subseteq G$. Suppose for contradiction that there exists some $\omega_3 \in G \setminus \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$.

Observe that $G \cap \mathbb{R}\omega_1 = \mathbb{Z}\omega_1$, because if there exists some $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ such that $\alpha\omega_1 \in G$ then $(\alpha - \lfloor\alpha\rfloor)\omega_1 \in G$ too; but $|(\alpha - \lfloor\alpha\rfloor)\omega_1| < |\omega_1|$, a contradiction to that $|\omega_1|$ was minimal. It then follows that $\omega_2 \notin \mathbb{R}\omega_1$, so ω_1, ω_2 are $\mathbb{R}$-linearly independent, which means they span $\mathbb{C} \cong \mathbb{R}^2$, so we can represent

$$\omega_3 = \alpha\omega_1 + \beta\omega_2$$

for some $\alpha, \beta \in \mathbb{R}$. WLOG, $|\alpha|, |\beta| \leq \frac{1}{2}$ (one can add/subtract appropriate amounts of ω_1 and ω_2), but then

$$\begin{aligned} |\omega_3| &= |\alpha\omega_1 + \beta\omega_2| \\ &< |\alpha||\omega_1| + |\beta||\omega_2| \\ &\leq \frac{1}{2}\left(|\omega_1| + |\omega_2|\right), \end{aligned}$$

where the strict inequality is from that ω_1, ω_2 are $\mathbb{R}$-linearly independent. But the minimality of ω_1 and ω_2 implies that

$$|\omega_3| \geq |\omega_1|, |\omega_2| \Rightarrow |\omega_1| > |\omega_2|, |\omega_2| > |\omega_1|,$$

which is a contradiction. □

1.3. **Definition and properties of elliptic functions.** As motivated by Section 1.1, we are particularly interested in periodic structure of the second type, i.e., *doubly periodic*. The functions that satisfy such a periodic structure, with some analytic requirement, are called, aptly, *elliptic functions*. Let us first recall what a meromorphic function f on $\mathbb{C}$ is.

Definition 1.2. A function $f : G \to \hat{\mathbb{C}}$ on a domain $G \subset \mathbb{C}$ is *meromorphic* if

(1) The set $P_f = \{z \in G : f(z) = \infty\}$ is discrete in G.
(2) The restriction $f|_{G \setminus P_f} : G \setminus P_f \to \mathbb{C}$ is holomorphic.

Definition 1.3. A *lattice* Ω in $\mathbb{C}$ is

$$\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent.

Definition 1.4. An *elliptic function* with respect to the lattice Ω is a meromorphic function f on $\mathbb{C}$ with

$$f(z + \omega) \equiv f(z) \text{ for all } z \in \mathbb{C}, \omega \in \Omega.$$

The overall structure of elliptic functions with respect to a fixed Ω is quite straightforward; one can add, subtract, multiply and divide them point-wise.

Definition 1.5. The set F, when associated with the binary operations addition and multiplication $+, \times : F \times F \to F$ is a *field* if both operations are associative and commutative, there exists additive and multiplicative inverses and identities, and multiplication is distributed over addition.

Proposition 1.6. *The elliptic functions with respect to lattice Ω constitute a field $K(\Omega)$. Moreover, if $f \in K(\Omega)$ then $f' \in K(\Omega)$.*

Proof. It is clear that the set of elliptic functions $K(\Omega)$ satisfy the field conditions with point-wise operations while maintaining the periodicity condition. A particular note, though, is with regards to taking multiplicative inverses, which one might run into the trouble of dividing by 0 if one deals with *holomorphic* functions exclusively, but in this case elliptic functions are *meromorphic*, so the operation is still well-defined.

Furthermore, if $f \in K(\Omega)$ then for any $z \in \mathbb{C}, \omega \in \Omega$,

$$\begin{aligned} f'(z+\omega) &= \lim_{h\to 0} \frac{f(z+\omega+h) - f(z+\omega)}{h} \\ &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z) \end{aligned}$$

so $f' \in K(\Omega)$ too. □

So we have that all elliptic functions with respect to a fixed Ω form a field, let's investigate a specific one. Take a particular $f \in K(\Omega)$. The periodic structure of f allows us to just look at a small portion of $\mathbb{C}$ to determine its global character. Since ω_1 and ω_2 do not lie on a straight line through 0, they span a half-open parallelogram

$$P(\omega_1, \omega_2) = \{z = t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 < 1\}.$$

Then for all $z \in \mathbb{C}$, there exists some $\omega \in \Omega$ such that $(z - \omega) \in P$ and $f(z) = f((z-\omega)+\omega) = f(z-\omega)$. So the values that f takes in $\mathbb{C}$ are entirely decided on P! So it shouldn't be surprising that Liouville's theorem works well here, if f is holomorphic, given how "small" P is.

Proposition 1.7. *If f is elliptic and holomorphic, then it is constant.*

Proof. The range of f is

$$f(\mathbb{C}) = f(P) \subseteq f(\overline{P})$$

but $f(\overline{P})$ is the continuous image of a compact so is bounded, so f is bounded. Liouville's theorem then implies that f must be constant. □

So the interesting ones (that we're going to look at) have poles. From now on, we implicitly assume that f is non-constant (has poles) and is elliptic with respect to lattice $\Omega = \langle \omega_1, \omega_2 \rangle$ with $\mathrm{Im}(\omega_2/\omega_1) > 0$:

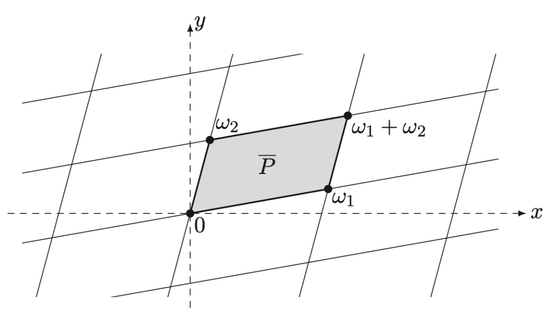

FIGURE 1. The lattice Ω and period parallelogram P. [1, page 175]

Remark 1.8. It has been seen that any value assumed by f is assumed in P, but that is also true for any $a + P := \{a + z : z \in P\}$ with some $a \in \mathbb{C}$, i.e., a translated version of P, with the same reasoning.

Proposition 1.9. *The number of poles of f in P is finite.*

Proof. Suppose not, then there exists a sequence of poles (z_n) in $P \subseteq \overline{P}$. $\overline{P}$ is compact so there exists a convergent subsequence of poles; but we know that the set of poles must be discrete. $\Rightarrow\Leftarrow$ □

Proposition 1.10. *Let f be elliptic with poles $z_1, \dots, z_n$ in P. Then*

$$\sum_{\nu=1}^{n} \operatorname{res}_{z_\nu} f = 0$$

Proof. WLOG, ∂P has no poles. If there are poles on ∂P, consider $a + P$ with small a instead. This consideration is valid since there are only finitely many poles. Then $\operatorname{int} P$ is a domain with boundary $\partial P = [0, \omega_1] + [\omega_1, \omega_1 + \omega_2] + [\omega_1 + \omega_2, \omega_2] + [\omega_2, 0]$, i.e., the directed perimeter of P. We can then apply residue theorem on $\operatorname{int} P$ to get

$$\begin{aligned} 2\pi i \sum_{\nu=1}^{n} \operatorname{res}_{z_\nu} f &= \int_{\partial P} f(z) \mathrm{d}z \\ &= \int_{[0,\omega_1]-[\omega_2,\omega_2+\omega_1]} f(z)\mathrm{d}z + \int_{[\omega_1,\omega_1+\omega_2]-[0,\omega_2]} f(z)\mathrm{d}z \\ &= 0 \end{aligned} \tag{1.11}$$

since f is ω_1- and ω_2-periodic. □

Corollary 1.12. *f has at least 2 poles in P, counting multiplicity.*

Proof. Suppose not, that f has 1 simple pole z_1 in P, but that would make

$$0 = \operatorname{res}_{z_1} f \neq 0$$

since z_1 is a simple pole. □

An interesting interplay of the closed-under-differentiation property of $K(\Omega)$ with the argument principle yields us the following result, that gives some impression of (admittedly not that relevant) Casorati-Weierstrass/Little Picard/Big Picard.

Proposition 1.13. *f assumes every $w \in \hat{\mathbb{C}}$ equally often.*

Proof. Fix $w \in \hat{\mathbb{C}}$. From the argument principle we know that for some domain G with positive boundary such that $f \neq w, \infty$ on ∂G, we have that

$$\frac{1}{2\pi i} \int_{\partial G} \frac{f'(z)}{f(z) - w} = N(w) - N(\infty) \tag{1.14}$$

where $N(w) = \#\{z \in G : f(z) = w\}$ and correspondingly for $N(\infty)$. It remains for us to choose a smart G — we just have to choose $G = a + P$ (possibly P) such that f doesn't attain w, ∞ on $\partial(a + P)$. But $f \in K(\Omega)$ so $\frac{f'}{f-w} \in K(\Omega)$, then a similar computation as (1.11) yields that LHS $= 0$, so $N(\omega) = N(\infty)$, which is just the number of poles of f in $a + P$, i.e., in P.

Therefore in P, f assumes every $w \in \hat{\mathbb{C}}$ equally often, in particular, equally number-of-poles often. □

This number is obviously essential to f, so we assign that as its *order*, just like how we assign some *degree* to polynomials and rational functions.

Definition 1.15. The *order* of $f \in K(\Omega)$ is the number of poles it has in P.

We've therefore said something about the *number* of inputs that gives f a fixed value, but we can also say something directly about the *value* of those inputs as well, in particular, for zeros and poles.

Proposition 1.16. *Let $\{a_\mu\}_{\mu\in[k]}$ be zeros with corresponding multiplicities $\{m_\mu\}$, and $\{b_\nu\}_{\nu\in[l]}$ be poles with multiplicities $\{n_\nu\}$ then*

$$\sum_{\mu=1}^{k} m_\mu a_\mu - \sum_{\nu=1}^{l} n_\nu b_\nu \in \Omega$$

Proof. The overview of this proof is that we want to say something about the value of the zeros and poles (a_μ, b_ν) and their corresponding multiplicities, so we can't simply use something of the form $\frac{f'}{f}$ again, to then only get the multiplicity. It's here that we are inclined to also multiply by z so that we get a_μ, b_ν as well. In view of this, we can define

$$g(z) = z\frac{f'(z)}{f(z)}. \tag{1.17}$$

Again, WLOG, $a_\mu, b_\nu \notin \partial P$. Then by residue theorem, we get that

$$\frac{1}{2\pi i}\int_{\partial P} g(z)\mathrm{d}z = \sum_{\mu=1}^{k} \mathrm{res}_{a_\mu} g + \sum_{\nu=1}^{l} \mathrm{res}_{b_\nu} g$$

We can then compute the residues for g, since z is holomorphic in P, we get that

$$\mathrm{res}_{a_\mu} g = (z\mid_{a_\mu})\mathrm{res}_{a_\mu}\left(\frac{f'}{f}\right) = a_\mu m_\mu$$

and

$$\mathrm{res}_{b_\nu} g = (z\mid_{b_\nu})\mathrm{res}_{b_\nu}\left(\frac{f'}{f}\right) = -b_\nu n_\nu$$

so (1.17) implies

$$\sum_{\mu=1}^{k} m_\mu a_\mu - \sum_{\nu=1}^{l} n_\nu b_\nu = \frac{1}{2\pi i}\int_{\partial P} g(z)\mathrm{d}z, \tag{1.18}$$

and our strategy to include z in $g(z)$ has paid off. It remains for us to show that RHS $\in \Omega$.

Consider the opposite sides $[0, \omega_1]$ and $[\omega_1 + \omega_2, \omega_2]$, which are offset by ω_2 in opposite directions. Then we have

$$\begin{aligned} g(z) - g(z+\omega_2) &= \frac{zf'(z)}{f(z)} - \frac{(z+\omega_2)f'(z+\omega_2)}{f(z+\omega_2)} \\ &= -\omega_2\frac{f'(z)}{f(z)} \end{aligned}$$

so

$$\begin{aligned}\int_{[0,\omega_1]+[\omega_1+\omega_2,\omega_2]} g(z)\mathrm{d}z &= -\int_{[0,\omega_1]} \omega_2 \frac{f'(z)}{f(z)}\\ &= -\omega_2 \int_{[0,\omega_1]} \frac{f'(z)}{f(z)}\\ &= -\omega_2 \int_\gamma \frac{1}{\zeta}\mathrm{d}\zeta\\ \Rightarrow \frac{1}{2\pi i}\int_{[0,\omega_1]+[\omega_1+\omega_2,\omega_2]} g(z)\mathrm{d}z &= \omega_2 \mathrm{wn}(\gamma,0) \in \mathbb{Z}\omega_2\end{aligned}$$

where γ is a closed curve "from" $\frac{1}{f(0)}$ "to" $\frac{1}{f(\omega_1)} = \frac{1}{f(0)}$. Similarly, we have that

$$\frac{1}{2\pi i}\int_{[\omega_1,\omega_1+\omega_2]+[\omega_2,0]} g(z)\mathrm{d}z \in \mathbb{Z}\omega_1$$

so it follows from (1.18) that

$$\sum_{\mu=1}^{k} m_\mu a_\mu - \sum_{\nu=1}^{l} n_\nu b_\nu = \frac{1}{2\pi i}\int_{\partial P} g(z)\mathrm{d}z \in \mathbb{Z}\omega_1 + 2\pi i\mathbb{Z}\omega_2 = \Omega.$$

□

2. The construction of elliptic functions

2.1. **The bottom-up approach.** We've discussed long and hard on properties of elliptic functions — let's construct some. Our approach for this section will be bottom-up, i.e., we try to build functions that are elliptic "out of the box". The most obvious candidate would be something of the form $f(z) = \sum_{\omega\in\Omega} F(z-\omega)$, which iterates through the entire lattice space to ensure periodicity of f with some F:

$$f(z+\omega) = \sum_{\omega\in\Omega} F((z+\omega)-\omega) = \sum_{\omega\in\Omega} F(z) = \sum_{\omega\in\Omega} F(z-\omega) = f(z).$$

We're looking for a nice enough — meromorphic in particular — f, so either $F(z) = z^k$ or z^{-k} come to mind; but if $F(z) = z^k$ then $f(z) = \sum_{\omega\in\Omega}(z-\omega)^k$ (if defined/converges at all) would just be holomorphic and thus constant. Therefore the next natural choice would be to use $(z-\omega)^{-k}$, for $k \geq 2$ since f has at least 2 poles (counting multiplicity) in P.

Proposition 2.1. *For $k \in \mathbb{N}$, define*

$$f_k(z) = \sum_{\omega\in\Omega} \frac{1}{(z-\omega)^k}, \tag{2.2}$$

then for $k \geq 3$, the series converge locally absolutely uniformly.

The proof of which depends on

Proposition 2.3. *The series $\sum_{\omega\in\Omega\setminus\{0\}} \omega^{-k}$ converges absolutely iff $k > 2$.*

Proof. (of Proposition 2.3) Essentially, we want to show that the quick ($k > 2$) rate of decay of ω enables the sum when taken over the entire lattice to converge absolutely, but $k \leq 2$ is not going to be quick enough.

For $l \in \mathbb{N}$, denote P_l as the parallelogram with vertices $\pm l\omega_1 \pm l\omega_2$. Then ∂P_l has $8l$ lattice points, and each lattice point $m\omega_1 + n\omega_2$ lies on $\partial P_{\max(m,n)}$. Then if we denote $\delta = dist(\partial P_1, 0) > 0$, then $dist(\partial P_l, 0) = l\delta$ by scaling.

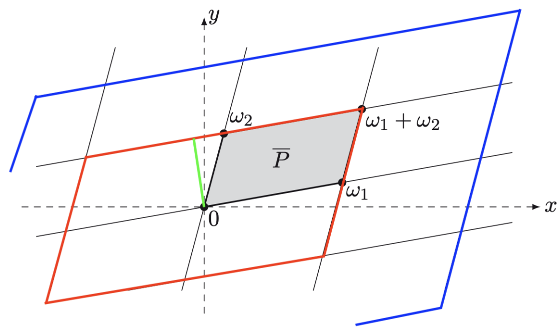

FIGURE 2. ∂P_1 in red, ∂P_2 in blue, and δ in green.

It then follows that

$$\sum_{\omega \in \Omega \setminus \{0\}} |\omega|^{-k} \leq \sum_{l=1}^{\infty} 8l(l\delta)^{-k} = 8\delta^{-k} \sum_{l=1}^{\infty} \frac{1}{l^{k-1}} < \infty$$

for $k > 2$.

On the other hand, when $k \leq 2$, we can make the same estimation with $\delta = \sup\{|z| : z \in \partial P_1\} > 0$, then

$$\sum_{w \in \Omega \setminus \{0\}} |\omega|^{-k} \geq \sum_{l=1}^{\infty} 8l(l\delta)^{-k} = 8\delta^{-k} \sum_{l=1}^{\infty} \frac{1}{l^{k-1}} = \infty$$

as required. □

Proof. (of Proposition 2.1) For each $|z| \leq R$, use the triangle inequality to lower bound $|z - \omega| \geq \frac{|\omega|}{2}$ all for sufficiently large $|\omega|$, hence upper bounding the $|z - \omega|^{-k}$ series with $|\omega|^{-k}$. See [1, page 178]. □

We said that our building blocks start with $k \geq 2$, but the proposition shows that f_k converges when $k \geq 3$. What happens when $k = 2$? Well, f_2 doesn't converge so we are unable to construct an elliptic function of order 2 with our current building blocks. However, we can construct one by integrating f_3 (bringing the powers from -3 to -2) along a path from 0 to z to yield a primitive. However, before we can do this, there are several things to check.

We can first leave aside z^{-3}, since this term is well-understood and its primitive on $\mathbb{C} \setminus \Omega$ is $\frac{-1}{2} z^{-2}$. Then for the rest, $\sum_{\omega \in \Omega \setminus \{0\}} (z - \omega)^{-3}$ is holomorphic on $\mathbb{C} \setminus \Omega \cup \{0\}$ with zero residues on $\omega \in \Omega \setminus \{0\}$ (around a fixed ω', the expansion is $(z - \omega')^{-3} + \sum_{\omega \in \Omega \setminus \{0, \omega'\}} (z - \omega)^{-3}$ so the coefficient of z^{-1} is 0, so residue is zero). It then follows that for any closed curve γ in $\mathbb{C} \setminus \Omega \cup \{0\}$, the residue theorem yields that the integral over γ is 0. Using this fact, the integral from a fixed point 0 to z is unambiguous and is indeed a primitive. Then we have that, for $\omega \in \Omega \setminus \{0\}$,

$$\begin{aligned}\int_0^z (\zeta - \omega)^{-3} \mathrm{d}\zeta &= \frac{-1}{2}\left[(\zeta-\omega)^{-2}\right]_0^z \\ &= \frac{-1}{2}((z-\omega)^{-2} - \omega^{-2})\end{aligned}$$

so we can write, combining with the term when $\omega = 0$,

$$g(z) = \frac{-1}{2}\left[\frac{1}{z^2} + \sum_{\omega \in \Omega\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)\right]$$

is a primitive of f_3 on $\mathbb{C} \setminus \Omega$.

Directly, g is meromorphic on $\mathbb{C}$, with the lattice points of Ω as its poles of order 2 and looks somewhat periodic, but it would be mistaken to immediately admit that $g \in K(\Omega)$. After all, we previously mentioned that $K(\Omega)$ is closed under taking differentiation, but is not guaranteed to be so under integration,. What we do know is that from the periodicity of f_3, for all $\omega \in \Omega$, we get

$$g'(z+\omega) - g'(z) \equiv 0,$$

so $g(z+\omega) - g(z) \equiv C(\omega) \in \mathbb{C}$, some constant dependent on ω.

Notice that g is an even function, so for $j = 1, 2$, we have

$$C(\omega_j) = g(\omega_j/2) - g(-\omega_j/2) = 0$$

so $C(\omega_j) = 0$. This implies $C(\omega) = 0$ for all $\omega \in \Omega$, simply because, say, for $\omega = m\omega_1 + n\omega_2$ with $m > 0$,

$$\begin{aligned}C(m\omega_1 + n\omega_2) &= g(z + m\omega_1 + n\omega_2) - g(z) \\ &= g(z + m\omega_1 + n\omega_2) - g(z + (m-1)\omega_1 + n\omega_2) \\ &+ g(z + (m-1)\omega_1 + n\omega_2) - g(z) \\ &= C(\omega_1) + C((m-1)\omega_1 + n\omega_2)\end{aligned}$$

and so on. In other words, we get that for all $\omega \in \Omega$,

$$g(z+w) - g(z) \equiv 0,$$

so indeed g is elliptic with respect to Ω. This g is special in ways that will be shown in later sections; so special that, it (up to removing an ugly constant) takes the name of a special mathematician, Karl Weierstrass (1815–1897).

Definition 2.4. The *Weierstrass $\wp$-function* of the lattice Ω is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right). \tag{2.5}$$

By construction, $\wp$ is an *even, elliptic* function of order 2 with double poles at lattice points. Its derivative, which is essentially a rescaled f_3, is

$$\wp'(z) = -2 \sum_{\omega \in \Omega} (z-w)^{-3},$$

which is an *odd, elliptic* function of order 3 with triple poles at lattice points, also by construction.

It is said that $\wp$ is the simplest elliptic function in some sense — in what sense? Well, in the order sense. $\wp$ achieves the minimum possible order for an elliptic function. Its derivative $\wp'$, having order 3, is next on the line. One might already expect that these 2 basic blocks can build up to all other elliptic functions. And that is true, with their even-odd dynamic.

Theorem 2.6. *(1) Every even elliptic function f is a rational function of $\wp$.*

(2) Every odd elliptic function g can be written as $g = \wp' R(\wp)$ where $R(\wp)$ is a rational function of $\wp$.

Lemma 2.7. *Let f_1 be an even elliptic function with poles (if any) in the lattice points. Then f_1 is a polynomial in $\wp$:*

$$f_1 = a_0 + a_1\wp + \cdots + a_n\wp^n$$

with $a_\mu \in \mathbb{C}$.

Proof. (of Lemma 2.7) The idea of the proof is quite simple. We try to compose f using powers of $\wp$ from the most negative power of z onwards.

If f is constant then we're done. If not, consider the Laurent expansion of f about the lattice point 0. Since f is even, there are only even powers of z:

$$f(z) = b_{-2n}z^{-2n} + \cdots, \quad b_{-2n} \neq 0$$

In particular, from the expansion, f is elliptic of order $-2n$. The Laurent expansion of $\wp$ around 0 is (from (2.5)) $z^{-2} + \ldots$, so

$$f_1(z) = f(z) - b_{-2n}(\wp(z))^n$$

would be an even, elliptic function of order $-(2n-2)$, also with poles in at most the lattice points. Performing induction in the order of elliptic f would yield the end of the proof. □

Proof. (of Theorem 2.6) *(1)* Given an even and elliptic f. Suppose that f has poles $z_1, \ldots, z_n$ in $P \setminus \Omega = P \setminus \{0\}$. Let us then try to "remove" these poles, just like how we can make a singularity z_0 of a meromorphic function removable by multiplying by some $(z - z_0)^k$ term.

Then for every $j \in [n]$, $[\wp(z) - \wp(z_j)] \mid_{z=z_j} = 0$, so there exists some $m_j \in \mathbb{N}$, at most the order of the pole z_j (to be exact, it is the ceiling of the quotient of the order of pole z_j to the order of z_j as a zero in $\wp(z)-\wp(z_j)$), such that $(\wp(z)-\wp(z_j))^{m_j} f(z)$ has a removable singularity at z_j. Perform the same procedure for all other poles in $P \setminus \{0\}$, we get that

$$f_1(z) = \prod_{j=1}^{k} (\wp(z) - \wp(z_j))^{m_j} f(z)$$

has no poles in $P \setminus \{0\}$, but can very well have a pole at 0. However, applying Lemma 2.7 yields that f_1 is some polynomial in $\wp$, so it follows that

$$f(z) = \frac{f_1(z)}{\prod_{j=1}^{k} (\wp(z) - \wp(z_j))^{m_j}}$$

is a rational function of $\wp$ (since $\wp(z_j)$ are just numbers).

(2) Apply *(1)* to $\frac{g}{\wp'}$, it is even and elliptic. □

So this is a super interesting phenomenon, that $\wp$ and $\wp'$ generate the entire field of $K(\Omega)$, since every function can be decomposed into even and odd parts. In fact, one can glean even more insight on the $\wp$ function by applying the crucial inductive procedure in Lemma 2.7 on the even elliptic function $(\wp')^2$ to get an expression between $\wp$ and its derivative as follows.

Proposition 2.8. *$\wp$ satisfies the differential equation*

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \tag{2.9}$$

where

$$g_2 = g_2(\Omega) = 60 \sum_{\omega\in\Omega\setminus\{0\}} \omega^{-4}, \quad g_3 = g_3(\Omega) = 140 \sum_{\omega\in\Omega\setminus\{0\}} \omega^{-6}.$$

Proof. See [1, page 181] □

2.2. **The top-down approach.** One now begs the question if instead, we are not allowed to construct just any elliptic function (with respect to Ω) and have to construct one that has prescribed zeros and poles (with corresponding multiplicities) on P. Can we do it? It turns out that the answer is yes, and the overarching idea is to show that it is sufficient for a function to satisfy the properties in Proposition 1.13 and Proposition 1.16, and then construct a particular (luckily, not-too-unnatural) function that does so. See [1, page 183].

3. Addition on plane cubics from elliptic functions

Let us switch gears a little bit; this section will aim to illuminate an addition structure that exists on plane cubics, using the differential equation that we've demonstrated for $\wp$ in Proposition 2.8. As before, fix a lattice Ω in $\mathbb{C}$. Then the corresponding $\wp$ and $\wp'$ only have poles at the Ω lattice points, so is holomorphic on $\mathbb{C}\setminus\Omega$. It follows that the map

$$\varphi : \mathbb{C}\setminus\Omega \to \mathbb{C}^2$$
$$z \mapsto (\wp(z), \wp'(z))$$

is also holomorphic. The link that connects elliptic functions to plane cubics is the differential equation (2.9), so the image of φ is contained in

$$E = \{(u, v) \in \mathbb{C}^2 : v^2 = 4u^3 - g_2u - g_3\} \tag{3.1}$$

where $g_2 = g_2(\Omega), g_3 = g_3(\Omega)$; the equation $v^2 = 4u^3 - g_2u - g_3$ is said to be in *Weierstrass normal form.* In fact φ is onto E, since $\wp$ is onto $\mathbb{C}$ (Proposition 1.13), and then we can match the first coordinate accordingly.

To do addition on E, we want to use the addition structure in $\mathbb{C}$ and then transfer it through φ over to E, but for now φ is only defined on $\mathbb{C}\setminus\Omega$. Therefore we'd like to extend it to all of $\mathbb{C}$, specifically to map Ω to something. The formula of $\wp$ and $\wp'$ suggests that we map to something like "infinity" — which is why we are going to embed $\mathbb{C}^2$ into the complex projective plane $\mathbb{P}^2(\mathbb{C})$.

Definition 3.2. $\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus \{0\})/_\sim$ where the equivalence relation on the RHS is $\mathbf{x} \sim \mathbf{y}$ if one is a $\mathbb{C}^*$-multiple of the other. As a side note, geometrically, the complex projective plane $\mathbb{P}^2(\mathbb{C})$ extends the notions of points and lines, so that there is a unique point that every 2 distinct lines intersect at, and there is a unique line that every 2 distinct points lie on.

We identify each point of $\mathbb{P}^2(\mathbb{C})$ with $[w_0 : w_1 : w_2]$, which capture the equivalence class and are called *homogenous coordinates* since $[w_0 : w_1 : w_2] = [\lambda w_0 : \lambda w_1 : \lambda w_2]$ for $\lambda \neq 0$. The natural embedding of $(u, v) \in \mathbb{C}^2$ into $\mathbb{P}^2(\mathbb{C})$ would then be $[1 : u : v]$, while the line at infinity that goes through all the points at infinity is $\{[0 : \cdot : \cdot]\}$.
The form of the equation on E is, then, in homogeneous coordinates:

$$\left(\frac{w_2}{w_0}\right)^2 = 4\left(\frac{w_1}{w_0}\right)^3 - g_2\left(\frac{w_1}{w_0}\right) - g_3$$
$$\Rightarrow w_0 w_2^2 = 4w_1^3 - g_2 w_1 w_0^2 - g_3 w_0^3,$$

so now we can find which "point at infinity" also satisfy this requirement. Plugging in $w_0 = 0$, we get that $w_1 = 0$, so the only other point in $\mathbb{P}^2(\mathbb{C})$ that satisfies this requirement is

$$P_0 = [0 : 0 : 1],$$

since $[0 : 0 : 0] \notin \mathbb{P}^2(\mathbb{C})$. This point will enable us to extend φ to $\mathbb{C}$, by mapping Ω to this new point!
Denote $\overline{E} = E \cup \{P_0\}$ where E is already embedded into $\mathbb{P}^2(\mathbb{C})$, then we can officially extend φ:

$$\begin{aligned} \varphi : \mathbb{C} \setminus \Omega &\to E \\ \Omega &\to P_0. \end{aligned}$$

Here is the crucial part. Due to its construction, φ inherits the periodic property of $\wp$ and $\wp'$. In particular,

Proposition 3.3.

$$\varphi(z_1) = \varphi(z_2) \Leftrightarrow z_1 - z_2 \in \Omega.$$

Proof. $\varphi(z_1) = \varphi(z_2) \Rightarrow \wp(z_1) = \wp(z_2)$ which implies either $z_1 - z_2 \in \Omega$ or $z_1 + z_2 \in \Omega$. If $z_1 + z_2 \in \Omega$ then $\wp'(z_1) = \wp'(z_2) = -\wp'(z_1)$ since $\wp$ is odd, so $\wp'(z_1) = \wp'(z_2) = 0 = \wp(-z_2)$, so $z_1 - z_2 \in \Omega$ by Proposition 1.16. □

It then follows that φ induces a bijection between the quotient group $\mathbb{C}/\Omega$ and $\overline{E}$ (each equivalence class $\mathbb{C}/\Omega$ (within each class, numbers differ by some $\omega \in \Omega$) is assigned to a particular member of $\overline{E}$); this is where the addition structure can be transferred over, as a *group homomorphism* of sort (a priori, $\overline{E}$ doesn't have an additive structure, but we're endowing it with one). Given $A, B \in \overline{E}$, then one can define

$$A + B \coloneqq \varphi(\varphi^{-1}(A) + \varphi^{-1}(B)).$$

So far this seems quite artificial, but we will arrive at a nice geometric interpretation of this addition structure later on. To find what this seemingly artifical addition looks like, let us first consider the case when $\varphi^{-1}(A), \varphi^{-1}(B)$ and $\varphi^{-1}(A) + \varphi^{-1}(B)$ are not on Ω.

Proposition 3.4. *Let $z_1, z_2, z_3 = z_1 + z_2 \notin \Omega$. Let*

$$A = (p_1, p_1') \coloneqq \varphi(z_1), B = (p_2, p_2') \coloneqq \varphi(z_2), C' = (p_3, -p_3') \coloneqq (\wp(z_3), -\wp'(z_3))$$

then A, B, C' are the intersection points of E with the unique line $L = \{(u, v) : v = au + b\}$ that goes through A, B.

Proof. WLOG, let $z_1, z_2 \in P$. Then consider

$$f(z) = \wp'(z) - a\wp(z) - b,$$

with a, b chosen such that $f(z_1) = f(z_2) = 0$:

$$a = \frac{p_1' - p_2'}{p_1 - p_2}, \quad b = p_1' - ap_1,$$

then f is elliptic of order 3. This looks complicated, but if we take a closer look at f and what it means for f to be zero, we see that on $\mathbb{C}^2$, the a and b constants are chosen such that $\varphi(z_1), \varphi(z_2)$ lie on the line $L_1 = \{(u, v) : v = au + b\}$, i.e., L_1 goes through A and B.

Back to f, then $f = \wp' - a\wp - b$ is of order 3 and has 2 zeros so far. Therefore, by Proposition 1.13 and Proposition 1.16, there exists a third zero $z_3' \in P$ and $z_1 + z_2 + z_3' \in \Omega$, i.e.

$$f(z_3') = 0 \Rightarrow f(z_3' - (z_1 + z_2 + z_3)) = 0 \Rightarrow f(-z_3) = 0,$$

so

$$\wp'(-z_3) = a\wp(-z_3) + b$$

but $\wp'$ is odd and $\wp$ is even so

$$-\wp'(z_3) = a\wp(z_3) + b \Rightarrow -p_3' = ap_3 + b.$$

In summary, we have

$$\begin{cases} p_1' = ap_1 + b \\ p_2' = ap_2 + b \\ -p_3' = ap_3 + b \end{cases}$$

so indeed $A, B, C' \in L_1$. □

Remark 3.5. If $p_1 = p_2$, i.e., $A = B$, the same proposition holds with

$$a = \frac{12p_1^2 - g_2}{2p_1'}, \quad b = p_1' - ap_1.$$

Proposition 3.6. *From Proposition 3.4 and Remark 3.5, we can now describe the geometric interpretation of the addition*

$$A + B = \varphi(\varphi^{-1}(A) + \varphi^{-1}(B)).$$

in 2 steps:

(1) *Given $A = (p_1, p_1'), B = (p_2, p_2') \in \overline{E}$, draw the unique line L_1 that goes through them. L_1 -•- A, B; and $L_1 \cap \overline{E}$ at $C' = (p_3, -p_3')$.*
(2) *"Take the negative" of the second coordinate to get (p_3, p_3') from $(p_3, -p_3')$, but what this means more generally in the geometric world is to take a detour to $\mathbb{P}^2(\mathbb{C})$, then C is simply the intersection of the line $L_2 : w_1 = p_3 w_0$ (vertical line in planar world) that goes through P_0 and C' with $\overline{E}$. In short, draw L_2 -•- P_0, C'; and $L_2 \cap \overline{E}$ at $C = A + B$.*

Remark 3.7. A complex line in $\mathbb{P}^2(\mathbb{C})$ intersects $\overline{E}$ in 3 points, counting multiplicity, so a simple point of tangency is counted twice, and an inflexion point thrice. As in the procudure above, P_0, C', C are the 3 points that L_2 intersects with $\overline{E}$. This plays into our "line-taking" procedure, where for example, if $A = B$, the line L_1 that we need to take would be the line tangential to $\overline{E}$ at $A = B$.

Remark 3.8. We have only done the analytic work for $A, B, A+B \neq P_0$, but in the step procedure above we've also included P_0 as a viable choice for A or B (in $\overline{E}$), because this geometric interpretation can be extended to all of $\overline{E}$.

Visually, we can refer to the figure below.

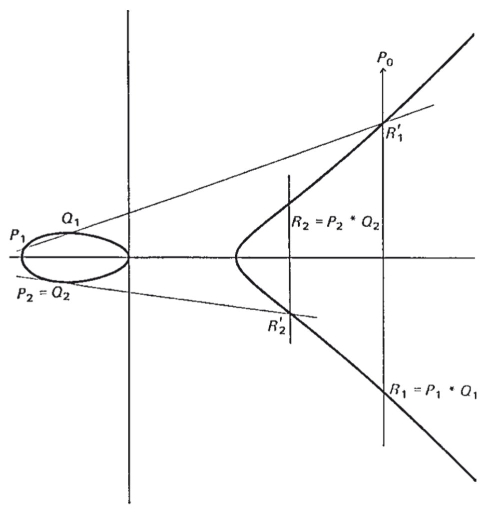

FIGURE 3. Addition on plane cubic. [1, page 191]

The figure illustrates 2 cases. For the sake of illustration, let us re-narrate the procedure in the context of the figure. For $P_1 + Q_1$, take the line that goes through them, this is L_1, then $R_1' \in L \cap \overline{E}$ is the point of intersection. Then the line L_2 to P_0 is simply the vertical line to "infinity", and we get $R_1 \in L_1 \cap \overline{E}$ at the bottom-right is $P_1 + Q_1$. For $P_2 + Q_2$, since $P_2 = Q_2$, L_1 is the tangential line to $\overline{E}$, and we get $R_2' \in L_1 \cap \overline{E}$. Draw a similar L_2 to the case before, and we get $R_2 = P_2 + Q_2$.

We've therefore endowed $\overline{E}$ with an addition structure, and making it an Abelian group with P_0 as its identity element (one can easily check, simply through the procedure). This is always possible, when the *discriminant* of the plane cubic E, $\Delta = g_2^3 - 27g_3^2 \neq 0$, i.e., E is *non-singular.* Simply put, the discriminant of the plane cubic is similar to the discriminant of a quadratic, in that it signifies information about the roots of the cubic.

As we conclude, notice that the geometric description that we give to addition on $\overline{E}$ relieves its parametrization of $\varphi : z \mapsto (\wp(z), \wp'(z))$, as well as its addition as corresponding to $A + B = \varphi(\varphi^{-1}(A) + \varphi^{-1}(B))$; it is purely geometrical. As it turns out, any $P_0 \in \overline{E}$, not just $[0:0:1]$, would suffice as the additive identity.

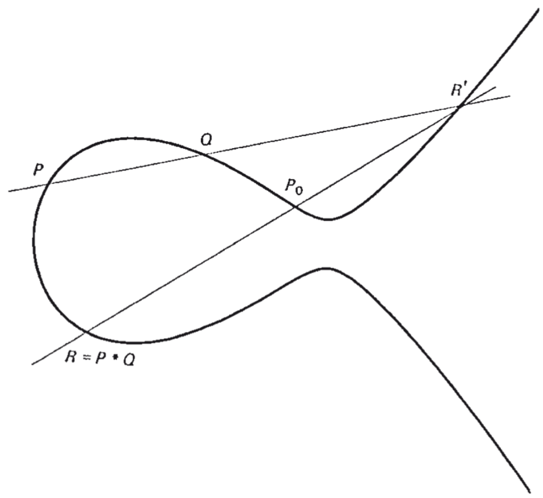

FIGURE 4. Arbitrary P_0. [1, page 192]

The above figure demonstrates our last remark, where P_0 is simply chosen as a point on E itself, and the procedure to get R' through L_1 is the same, while getting R is slightly different: it is no longer drawing the vertical line L_2 through ∞, but it is still drawing through P_0, and we get R at the bottom-left corner.

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[1] Wolfgang Fischer, Ingo Lieb. A Course in Complex Analysis: From Basic Results to Advanced Topics. 2010.