In response to: "What role did philosophy and/or philosophical debates play in the development of modern European science?"

Russell's Principia and Hilbert's Program: Mathematics after Kant.

In this essay, I will attempt to illustrate how math influenced philosophy in the 19th century, and how this change in philosophy in turn influenced mathematics in the early 20th century. In particular, I first make the argument that developments in non-Euclidean geometry and the rigorization of analysis (or colloquially, calculus) relieved mathematics of the role of an intuition-based study of quantity, and expanded what a philosophy of mathematics can be. Second, I trace later frameworks in the foundations of mathematics, as exemplified by the titular Russell's *Principia Mathematica* and Hilbert's Program, back to these philosophical roots, one of logicism and the other formalism.

To talk about what it became, we should first talk about what it had been. Immanuel Kant, specifically his *Critique of Pure Reason*, was so influential that many nineteenth-century arguments about the nature of mathematics can be seen as responses to Kant's philosophy: extensions of it, reinterpretations of it, breaks with it, emancipation from it, rediscovery of it.[1] Thus in what follows, for comparison purposes, his framework will similarly only broadly serve as the representative of the then-predominant philosophical understanding of epistemology, i.e., *how* we know things, and consequentially, what the limit of things we can say about things are. I do not detail *why* Kant's line of thinking was so particularly pervasive and take it as granted.

In layman's terms, Kant's framework is as follows.[2] He first classifies the different kinds of *judgments* we make, statements with a subject and a predicate. The first dimension is whether a judgment is *analytic*, a judgment whose predicate is already contained in the subject, e.g., "All bachelors are unmarried" ("unmarried" is contained in "bachelors"), or *synthetic*, a judgment whose predicate is not already contained in the subject, and therefore

1. Jeremy Gray, *Plato's Ghost: The Modernist Transformation of Mathematics* (Princeton University Press, 2008), 78, accessed March 13, 2026, http://www.jstor.org/stable/j.ctt7rq1t.
2. My modest attempt to understand Kant.

adds new information to the subject, e.g., "The car is blue" ("blue" is not contained in "car".) Another dimension of classification is whether it is *a posteriori*, knowledge derived from (sensory) experience, or *a priori*, knowledge absolutely independent of experience and therefore universal and *necessary.* Understand *necessary* here as 'must be true and could not possibly be false, regardless of the realization of the world,' whereas *a posteriori* judgments hinge on the particular experience and are therefore only *contingent*, only true dependent on the world realization. We can then say a few things. First, all *a posteriori* judgments are *synthetic.* If a judgment is contingent on the world realization, then surely its predicate cannot have already been contained in the subject. Second, all *analytic* judgments are *a priori.* This is clear, because if the predicate is already contained in the subject, the judgment is necessarily true "by definition," independent of world realizations.

The only category we have not said anything about is whether *synthetic*, *a priori* judgments are possible. They are the crown jewels in this framework, for they are both reliable (*a priori*, necessarily true, independent of world realization) and cumulative (*synthetic*, non-*analytic*, non-obvious, actually saying something new). Kant's answer? A triumphant yes, with paradigm examples as judgments of mathematics: "All mathematical judgements, without exception, are synthetic."[3] How this is possible for Kant is via *pure intuition. Intuition* is a technical term in Kant's epistemology, and means a direct intellectual apprehension of an object.[4] It belongs to the perceiving mind and is passive; understand it as a "generalized sense." Kant then makes the distinction between the *empirical intuition* and the *pure intuition.* For me, the *empirical intuition* is the regular "sense" (visual, auditory, etc.) that is derived from experience, whereas the *pure intuition* is derived from the structure of experience. It is then through the *empirical intuition* that we can make *a posteriori* judgments, whereas through *pure intuition*, the prime examples of which are that of Space and Time, that we can make *synthetic, a priori* judgments. How? Kant first takes arithmetics as an example of the pure intuition of Time at work. It is successively adding up units across time,

3. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 80.
4. Gray, 80.

an act of the pure intuition of Time, that we come to realize that $7+5=12$. Another example is Euclidean geometry, where judgments are justified with the pure intuition of Space. The important claim is that these are *synthetic* judgments and not *analytic*, because they are constructed and seen in the pure intuition space. To realize $7+5=12$, one constructs this truth by enumerating, say, 7 and 5 objects in the space of the *pure intuition* of Time, and "see" that they are 12 objects altogether. And when one says the "straight line is the shortest curve between two points," one constructs this straight line in the space of the *pure intuition* of Space, and "see" this truth through this visualization, this act of intuiting.

The "Euclidean" descriptor here is of absolute essence for Kant, for he believed that human pure intuition of Space are encoded in Euclid's *Elements*, a mathematical treatise written by Euclid around 300BC of definitions, postulates, geometric constructions and proofs in plane geometry, solid geometry and elementary number theory. For Kant, such encoding meant that the geometric mathematics that followed are in congruence with his framework. More broadly, for many others at the time, the definitions and postulates as laid out by Euclid were also considered as obviously true in the physical world. Until the advent and acceptance of non-Euclidean geometry.

Non-Euclidean geometry grew out of the gripping fascination of mathematicians over millenia with proving the last postulate of Euclid's list of five for plane geometry (instead of taking it as a postulate). To see why, I briefly review the first four: 1. A straight line segment can be drawn between any two points, 2. Any straight line segment can be extended indefinitely in a straight line, 3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center, 4. All right angles are congruent. With that in mind, here is the fifth, also known as the Parallel Postulate: 5. If a straight line intersects two other straight lines, the two interior angles on the same side add to less than two right angles if and only if the two lines, if extended indefinitely, meet on that side. The fascination should now be evident: the fifth postulate, verbose and unintuitive, seems like something Euclid could not prove himself and had to take for granted; resolving it by showing it through the other four postulates would have completely changed the foundations of (Euclidean) geometry at the time.

János Bolyai (1802-1860) was one such mathematician.[5] His father, Farkas Bolyai was a professor of mathematics and was obsessed with the Parallel Postulate himself. The younger Bolyai took after his father and attempted to prove the postulate, however, after repeated failed attempts to prove it, Bolyai started floating the idea that his failures were due to that the postulate was simply not true, and pivoted to show the existence of a geometry independent of the postulate. To this, his father warned him: ”Learn from my example: I wanted to know about parallels, I remain ignorant, this has taken all the flowers of my life and all my time from me.”[6] Fortunately Bolyai persevered and some successes in this line of work were published in 1832, and a copy was sent to the renowned Carl Friedrich Gauss (1777-1855), to which Gauss replied that he could not commend Bolyai for the excellent work, because that would mean commending himself: Gauss had also been working on the same ideas. And as such was one of the many seeds for non-Euclidean geometries sown.

I give a quick, illustrative example of a non-Euclidean geometry: spherical geometry. As the name suggests, spherical geometry concerns itself with geometry on a sphere, say, the two-dimensional sphere $\mathbb{S}^2$, e.g., the Earth's surface (two-dimensional, because locally, it looks like a plane). While one can study $\mathbb{S}^2$ as embedded in the Euclidean three-dimensional space $\mathbb{R}^3$, one can simply ignore such an embedding and deal with the intrinsic geometry of the sphere. And analogous to how Euclidean geometry has points and straight lines as its basic concepts, spherical geometry has points and great circles. It is easily seen that a great circle is the shortest curve between two points on the sphere, which conforms to the intuition of the Euclidean “line,” but whereas two Euclidean lines can only intersect at at most one point, great circles intersect at two (antipodal) points.

Spherical geometry is representative of the leap it took to develop non-Euclidean geometries — it questions what “points” and “lines” mean at their core, and what “geometry” can

5. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 46.
6. Gray, 46.

mean. Another eminent figure whose thought was along these lines was Nikolai Lobachevsky (1792-1856), who viewed terms like "line," "surface," and "position" as obscure, and certainly not fundamental. Rather, he thought that geometry should be based on bodies and the motion of bodies,[7] and proposed an alternative geometry which would then end up as hyperbolic geometry, in which the angle sum of a triangle is always less than two right angles. The flourishing and acceptance of non-Euclidean geometry, as initially developed by Bolyai and Lobachevsky, was gradually fulfilled with the publications of Gauss's *Nachlass*, Bernhard Riemann's (1826-1866) *Habilitationsvortrag* and Eugenio Beltrami's (1835-1900) *Saggio* in the late 1860s. In particular, Riemann played an outsized role in creating modern, generalized frameworks of geometry (Riemannian manifolds and differential geometry) in which we simply start with a set of points and a notion of distance, so that we can talk about shortest paths between points and measure angles of crossing curves.[8] Not only is Euclidean geometry no longer the starting point, it is wholly irrelevant.

Let's bring it back to philosophy. The upshot is that these developments out of geometry and their growing public acceptance in the late 19$^{\text{th}}$ century meant that the invocation of the *pure intuition* of Space is no longer feasible. There is no longer one "Space" that is "intuitive," for there are many possible geometries. The *intuition* framework fails, both as a technical term in Kant's epistemology and the colloquial "intuition" for laymen and mathematicians alike. The rigorization of analysis, in particular the foundations of the real numbers and properties of functions on the real numbers, further illustrates the collapse of the intuition. The short narrative, given our limited space, is that through Newton and Leibniz's invention of calculus in the 17$^{\text{th}}$ and its development throughout 18$^{\text{th}}$ century, calculus with the vague notion of infinitesimals was not properly rigorized until Augustin-Louis Cauchy's (1789-1857) arithmetical treatment of these ideas. Cauchy provided the first rigorous formulation of the calculus that mathematicians then and since have been able to agree was adequate. He defined what it is for a function to be integrable, to be continuous, and to be differentiable,

7. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 45.
8. Gray, 52.

using careful limiting arguments.[9] A function's continuity (in an interval) essentially means being able to draw its graph without lifting one's pen, while differentiability (at a point) means being able to draw a tangent at that point. Throughout these developments, it was the prevalent intuition and acceptance that continuous functions should be differentiable everywhere, only except for kink-type points, like the graph of $y = |x|$ at $x = 0$. Once again, "intuition" failed us: Karl Weierstrass (1815-1897) later found the eponymous, fractal-like Weierstrass function that is continuous everywhere, but differentiable nowhere.

The collapse of the intuition caused the mathematics community to go on an intense soul-searching journey. By 1900, the nature of mathematical objects was contentious and obscure. The arithmetization of analysis broke the naive identification of mathematical with physical objects, and geometry was increasingly abstract. Not only were these shifts ontological, they were also epistemological, as the nature of proof and the relation of mathematics to logic became a matter of animated research.[10] As such, mathematics gradually fashioned itself a new image: from what had simply been a study of quantity grounded in intuition, it became autonomous, abstract, axiomatic, and unconstrained by applications, even to physics. Sociological factors were also at play. In many universities across Europe, there were separate departments for mathematics and physics, there were separate journals, and recognizable divisions existed between the subjects and between their practitioners. The more critical mathematicians were aware that they therefore had to base their claims for the quality and value of mathematics on more intrinsic grounds.[11] Furthermore, the professionalization of mathematics implicitly introduced standards to which its members are publicly held, and mathematicians were evaluated based on their arguments as laid out in journals and books, which required more than a touch of intuition, now absolute and abstract rigor.

We have therefore seen how the developments in late 19$^{\text{th}}$ century mathematics influenced the philosophy of mathematics. The philosophical shifts they induced allowed for a

9. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 62.
10. Gray, 271.
11. Gray, 271.

reimagination of what mathematics is, can be and can be done. In the early 20$^{\text{th}}$ century, Bertrand Russell's logicism and David Hilbert's formalism were two such reimaginations.

Bertrand Russell (1872-1970) identified with the logicist philosophy. He claimed that geometry, algebra, and presumably any other form of mathematics are identical with logic in the nature of the knowledge they offer. They are nothing more nor less than chains of logical inference, i.e., mathematics can only be *analytic* judgments, and that mathematics is synonymous with symbolic logic. In 1903, Russell opened his *The Principles of Mathematics* with the famous claim "Pure mathematics is the class of all propositions of the form 'p implies q'." Soon, this work would be extended in the early 1910s by his co-written work with Alfred Whitehead, *Principia Mathematica*, where they further reiterated on that mathematics is nothing more than symbolic logic, and provided the raw ingredients for mathematicians, the terms and methods of reasoning, in the symbolic-logical framework. In this work, infamously, it took them a mere 379 pages to symbolically show $1 + 1 = 2$. It would be amiss to mention the logicist movement without bringing up Gottlob Frege (1848-1925), whose work *Begriffsschrift* in 1879 laid out the characterizations of the movement, with the argument that a scientific proof is best apprehended through its proof, however it might have been discovered; thus psychological genesis has no role to play,[12] a direct refutation to Kant. In this work, he sought to show that arithmetics, and in later years all of mathematics, can be derived from logic, charting out the line of thinking Russell would later adopt.

In contrast to the logicists in Russell and Frege who believed that mathematics is logic and their truth can be traced to logical laws, David Hilbert (1862-1943) was a formalist, who saw mathematics as only a formal system of symbols and was more concerned with the consistency of axiom systems. In what follows, I will attempt to represent the formalism framework to the best of my limited ability. By 1904, Hilbert realized the difficulty in his attempt to capture the essential reason mathematics was correct is that the consistency of some axiom systems relied on the consistency of others, in particular that of geometrical

12. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 159.

axioms was reduced to the consistency of arithmetics. In other words, we can demonstrate that a collection of points, lines, planes, as defined by coordinates and equations which are themselves represented by numbers, satisfy the geometrical axioms, but that demonstration technique makes it even more crucial to show the consistency of the underlying representative system, i.e., numbers and arithmetics.[13] In his *Axiomatisches Denken* in 1918, Hilbert thus claimed that the theory of the integers and the theory of sets must both be axiomatized. To do this, nothing other than logic could be invoked, and therefore "it appears necessary to axiomatise logic itself and to prove that number theory and set theory are only parts of logic." In 1922, his *Neubegründung der Mathematik* further sketched out his "proof theory," an attempt to subject collections of proofs to mathematical and logical analysis, in a machine-like manner, where one can imagine first taking an axiom system and listing all the initial statements, then all derivable in a single step from those, then those in two steps, and so on.[14] Of course, Hilbert's machinery was more involved, but the upshot holds: that it remains his Program to then write down an axiom system that both generates set theory and number theory and is checkably consistent per the procedure above, because the consistency of sets and arithmetics lies as the foundation of all things else.

It will eventually turn out that Hilbert's Program is impossible (for reasons still incomprehensible to me) and that few to no mathematics undergraduates will pick up Russell's *Principia* to believe the validity of $1 + 1 = 2$. However, these early 20[th] century initiatives set in sequence further crises in the foundations of mathematics, logic and computability that were hitherto unconsidered, and their eventual resolution through the works of Gödel, Church, Turing, and others. They themselves are the consequence of the refutation by the development of mathematics against the Kantian perspective in the late 19[th] century, and constitute a new view of mathematics that is now autonomous, ever more abstract and no longer solidly grounded in the realm of sensible intuition.

13. Gray, *Plato's Ghost: The Modernist Transformation of Mathematics*, 410.
14. Gray, 411.